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Matrix representations of holomorphic curves on $\mathbb{T}_4$

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Abstract: We construct a matrix representation of compact membranes analytically embedded in complex tori. Brane configurations give rise, via Bergman quantization, to U(N) gauge fields on the dual torus, with almost-anti-self-dual field strength. The corresponding U(N) principal bundles are shown to be non trivial, with vanishing instanton number and first Chern class corresponding to the homology class of the membrane embedded in the original torus. In the course of the investigation, we show that the proposed quantization scheme naturally provides an associative star-product over the space of functions on the surface, for which we give an explicit and coordinate-invariant expression. This product can, in turn, be used the quantize, in the sense of deformation quantization, any symplectic manifold of dimension two.

Keywords: D-branes, M(atrix) Theories, Non-Commutative Geometry.

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Contents

1. Introduction

Matrix theory [2]–[5], [7, 8, 9] is believed to describe, in the limit of large N, the fundamental degrees of freedom of M-theory. In fact, within the same theory, both fundamental particles and extended objects are described in a unified way. It is indeed remarkable that one can start with a theory of gluons in a certain dimension (matrix theory is nothing but $9+1$ U(N) Super-Yang-Mills theory dimensionally reduced to $0+1$ dimensions) and describe, in a dual way, a seemingly unrelated theory of gravity in a different space-time dimension $(10+1)$. In light of this fact, it is important to better understand the relations that exist between the two points of view, and to precisely describe how to represent in matrix language objects which are familiar from the M-theory prospective, and which are described at low energies within 11-D super-gravity.

It is of interest, in particular, to consider matrix theory configurations which represent, within the gravity description, extended membranes. More specifically one can study BPS membrane states, which only partially break supersymmetry, and which are not expected, on general grounds, to be effected by quantum corrections. Within the framework of supermembrane theory [10, 11], one can show that the BPS condition is equivalent to the requirement that the brane be embedded holomorphically in space, and it is therefore natural to look for matrix representations of holomorphic curves.

In [1] the question of representation of holomorphic curves embedded in non-compact space was analyzed in detail. In particular the problem was rephrased as a problem in geometric quantization, with $\varepsilon \sim L_P^3/R$ playing the role of the Planck constant. A specific quantization scheme was proposed, based on the concept of Bergman projection, and the matrices representing the curve were taken to be operators acting on the infinite dimensional space of holomorphic functions living on the brane. In order to preserve the BPS character of the configuration, one needs to choose a specific inner product on the space of functions, which is related to a deformation of the Kähler potential of the brane. Using an explicit expansion for the Bergman projection, the deformation was determined asymptotically in ε.

In this paper we extend the results of [1] to the interesting case of holomorphic curves embedded in complex tori. The first major difference is that the branes can now be taken to be compact. This requires an extension of the quantization scheme proposed in [1], in which we take the underlying Hilbert space to be the finite-dimensional space of holomorphic sections of a specific line bundle over the brane. The second major difference with the basic case considered in [1] comes from the fact that, although the target space is still flat, we cannot quantize directly the coordinate functions, since they are multivalued on the membrane. We solve this problem using an extension of T-duality [6] appropriate to the present context, and we relate the brane configurations on the torus to U(N) Yang-Mills configurations on the dual torus. The resulting U(N) bundle is non trivial, even though it has vanishing instanton number. In fact the first Chern class of the bundle corresponds to the homology class of the membrane embedded in the original torus. Moreover the BPS character of the original membrane configuration is now translated in a dual condition for the corresponding U(N) gauge potential. More precisely, we show that the corresponding field strength $\mathcal{F}$ is almost-anti-self-dual, in the sense that

$$\begin{aligned}\mathcal{F}_{12} + \mathcal{F}_{34} &\sim \varepsilon\,,\\ \mathcal{F}_{13} - \mathcal{F}_{24} &= \mathcal{F}_{14} + \mathcal{F}_{23} = 0\,.\end{aligned}$$

We also show that the configurations described are stable for to topological reasons.

The basis of most of the discussion in this paper is the quantization scheme which is analyzed in detail in the first part of the paper. In particular a very important tool which is analyzed at length and used repeatedly, is a specific non-commutative product (called star-product and denoted with $\star$) between functions on the brane. We show that the product $\star$, which was introduced in [1], is an associative operation. In particular, if we recast our result in the language of deformation quantization [14]–[17], [19], we show that the formula for the star-product can be used to quantize any symplectic manifold of real dimension 2.

The structure of this paper is as follows: in section 2 we review the results of [1] and extend them to a more general setting, which is needed in the subsequent part of the discussion. In particular we introduce the general concept of quantization

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used throughout the paper, and we define a star product $\star$ on the space of functions. Section 3 is entirely devoted to show that the product $\star$ is actually associative. This fact is heavily used in sections 5 and 7, and connects our discussion to the theory of deformation quantization. Section 4 briefly describes how to go from a know star-product to a full quantization scheme, and this result is then used in section 5 to discuss general results on the quantization of holomorphic curves embedded in Kähler manifolds. Section 6 describes how to compute, asymptotically in the quantization parameter ε, traces of operators obtained using the quantization scheme described in section 2. Finally section 7 is devoted to the main result of this paper. Using the results of the previous sections, we show that we can associate to each holomorphic curve embedded in a complex torus an almost-anti-self-dual Yang-Mills configuration on the dual torus. We conclude the paper in section 8 with suggestions for future research.

2. From Bergman projections to star products

In this first section we are going to recall and extend some of the main results originally derived in [1]. In particular we are going to extend the results on Bergman projections and Bergman quantization [13, 1, 18] to a more general setting, which is needed to tackle the problem of representation of holomorphic curves embedded in compact spaces. Although the exposition of the main ideas will be self-contained, we will omit most of the proofs, since they are essentially identical to the ones examined in [1]. In what follows, we will try to adhere, as much as possible, to the notation of [1].

Let us consider a compact Riemann surface Σ of genus g, on which we fix an arbitrary holomorphic line bundle S. We will denote by K the canonical line bundle on Σ, and by T the product bundle

$$T = S^{-1} \otimes K \,.$$

Given two sections ϕ, ψ of S, we wish to define an inner product $\langle \phi | \psi \rangle$ by integrating $\overline{\phi}\psi$ on the surface Σ against a suitable measure. In order to do this covariantly, we need to fix, first of all, a real and positive section C of the line bundle $T \otimes \overline{T}$.[1] It is then clear that the measure

$$\Omega(z) = i\, C(z) \;\; dz \wedge d\overline{z}$$

transforms as a section of $S^{-1} \otimes \overline{S}^{-1}$, and that the expression $\overline{\phi}\psi\, \Omega$ represents a well-defined 2-form on the surface Σ. We can then define the inner-product $\langle \phi | \psi \rangle$ as

$$\langle \phi | \psi \rangle = \int_\Sigma \overline{\phi}\psi\, \Omega \,.$$

[1] Note that the conditions of reality and positivity of C are well defined, since the bundle $T \otimes \overline{T}$ has real and positive definite transition functions.

Following the notation of [1], we will denote with $\mathcal{V}$ the Hilbert space of sections of S (not necessarily holomorphic) which have finite norm with respect to the above inner product, and we shall call $\mathcal{H} \subset \mathcal{V}$ the subspace of $\mathcal{V}$ consisting of holomorphic sections. Finally we will let π be the orthogonal Bergman projection onto $\mathcal{H}$

$$\pi : \mathcal{V} \longrightarrow \mathcal{H}\,.$$

Let us note that the choice of section C not only provides us with a specific inner product on $\mathcal{V}$, but also gives us a connection and a covariant derivative for sections of specific line bundles. To be more exact, let us introduce the connection

$$\Gamma = \partial \ln C$$

and let us consider a section ϕ of $T^a \otimes$ a.h., where a.h. stands for an arbitrary anti-holomorphic line bundle. It is then easy to show that

$$\nabla\phi = (\partial - a\Gamma)\phi$$

is a well-defined section of $K \otimes T^a \otimes$ a.h.. A similar result also holds for anti-holomorphic covariant derivatives. Finally we note that the curvature of the connection just described is given by the $(1,1)$ tensor

$$R = \partial\overline{\partial} \ln C\,.$$

We now briefly summarize the properties of the projection π in the following

Claim 1 *The projection π defined above satisfies the following properties:*

1. *The projection π has an integral representation. More precisely, let h_i be an orthonormal basis for $\mathcal{H}$ and consider the Bergman kernel*

$$K(z,w) = \sum_i h_i(z)\overline{h}_i(w)\,.$$

The kernel K is a well-defined bi-section, independent of the choice of orthonormal basis h_i of $\mathcal{H}$. Moreover, if $\phi \in \mathcal{V}$, one has that

$$\pi(\phi)(z) = \int_\Sigma K(z,w)\phi(w)\,\Omega(w)\,.$$

2. *If $\phi \in \mathcal{H}$ is holomorphic , then $\pi(\phi) = \phi$.*

3. *For any $\phi \in \mathcal{V}$, one has that $\overline{\nabla}\pi(\phi) = 0$.*

4. *Let X be a section of T^{-1} such that $\nabla X \in \mathcal{V}$. Then $\pi(\nabla X) = 0$.*

5. *Let $X_1, \ldots, X_{n-1}$ be holomorphic $(-1,0)$ vector fields, and let X_n be a holomorphic section of T^{-1}. If ϕ is an analytic function, then*

$$\pi(\phi \nabla X_1 \cdots \nabla X_n) = (-1)^n X_n \cdots \nabla X_1 \nabla \phi\,.$$

The proof of the above claim is identical to the one given in [1], and we therefore refer the interested reader to that paper for further details.

One of the main results of [1] was to show that the projection π possesses not only an integral representation but also, in an asymptotic sense, a differential one. Again the argument is essentially identical in the present setting, so that we content ourselves to state the following

Claim 2 *Let $R = \partial\overline{\partial} \ln C$ be the curvature tensor. Construct a sequence P_n of $(1,1)$ tensors starting from*

$$P_1 = R$$

and using the recursion relation

$$P_n = P_{n-1} + P_1 + \partial\overline{\partial} \ln(P_1 \cdots P_{n-1})\,. \tag{2.1}$$

If $\phi \in \mathcal{V}$, we then have that

$$\pi(\phi) = \sum_{n=0}^{\infty} (-1)^n \nabla \frac{1}{P_1} \cdots \nabla \frac{1}{P_n} P_1 \cdots P_n \frac{1}{P_n} \overline{\nabla} \cdots \frac{1}{P_1} \overline{\nabla} \phi\,,$$

where

$$\nabla = \partial + \Gamma\,, \qquad \overline{\nabla} = \overline{\partial}\,.$$

To conclude this section, we follow the general philosophy of [1] and discuss the process of quantization [12]. Consider a generic function A on the surface Σ, and let ϕ be an element of $\mathcal{H}$. The section $A\phi$, obtained by multiplying pointwise the original section ϕ with the function A is not necessarily holomorphic. On the other hand we can extract the holomorphic part by acting with the projection π, thus obtaining the element $\pi(A\phi)$ in $\mathcal{H}$. This process assigns to each function A an operator on $\mathcal{H}$. More precisely, if we denote by $\mathfrak{G}$ the space of complex functions on Σ, we have constructed a quantization map $\mathcal{Q} : \mathfrak{G} \to \mathrm{End}(\mathcal{H})$ which associates to each function A the operator $\mathcal{A} = \mathcal{Q}(A)$ defined by

$$\mathcal{A}(\phi) = \pi(A\phi)\,.$$

The map $\mathcal{Q}$ is compatible with complex conjugation, since $\mathcal{Q}(\overline{A}) = \mathcal{Q}^\dagger(A)$. Moreover it respects the complex structure on Σ, since $\mathcal{Q}(A)\mathcal{Q}(B) = \mathcal{Q}(AB)$ whenever A and B are both holomorphic. Finally, using the asymtotic expansion for the Bergman projection π and following [1], we can show that

$$\mathcal{Q}(A)\mathcal{Q}(B) = \mathcal{Q}(A \star B)\,,$$

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where we have introduced, on the space of functions $\mathfrak{G}$, a product $\star$, called star product, given explicitly in terms of the tensors P_n by the formula

$$A \star B = \sum_{n=0}^{\infty} P_1 \cdots P_n \left(\frac{1}{P_n} \partial \cdots \frac{1}{P_1} \partial A \right) \left(\frac{1}{P_n} \overline{\partial} \cdots \frac{1}{P_1} \overline{\partial} B \right) . \qquad (2.2)$$

3. Star products: proof of associativity

In section 2 we introduced a specific star product $\star$ on the space $\mathfrak{G}$ of complex functions on the surface Σ. This section is entirely devoted to show that the product $\star$ is associative. This fact will be repeadetly used in the subsequent sections, and will be a key element in the basic construction of this paper described in section 7. In particular are able to consider $\mathfrak{G}$ as an associative algebra, and we can therefore use most of our intuition about operators on Hilbert spaces in the different context of functions over a given surface.

The proof of associativity is rather lenghty and technical, and it can be skipped at first reading, since nothing from the following discussion (aside from the result itself) will be used in later sections. On the other hand, before we start the proof, let me briefly connect our result to the literature on deformation quantization [14]–[17], [19], where similar products have been stiudied at length.

The theory of deformation quantization starts with a choice of a real manifold Σ, together with a Poisson structure — i.e. an antisymmetric tensor ω^{ij} which satisfies

$$\omega^{\alpha\delta} \partial_\delta \omega^{\beta\gamma} + \omega^{\beta\delta} \partial_\delta \omega^{\gamma\alpha} + \omega^{\gamma\delta} \partial_\delta \omega^{\alpha\beta} = 0 \, , \qquad (3.1)$$

(if $\det \omega \neq 0$, then the manifold is symplectic). One also considers the space $\mathfrak{G}[[\varepsilon]]$ of formal power series

$$A = A_0 + A_1 \varepsilon + A_2 \varepsilon^2 + \cdots ,$$

where the A_i are functions on the manifold Σ, and ε is a quantization parameter. Given two elements A and B in $\mathfrak{G}[[\varepsilon]]$, one may use ω^{ij} to define a Poisson bracket

$$\{A, B\} = \omega^{ij} \partial_i A \partial_j B \, ,$$

which satisfies the Jacobi identity thanks to (3.1). The main problem of deformation quantization in then to define, on the space $\mathfrak{G}[[\varepsilon]]$, an associative product $\star$

$$A \star B = \sum_{n=0}^{\infty} \varepsilon^n S_n(A, B) \qquad (3.2)$$

such that

1. The S_n are bilinear local functionals of A, B (S_n depends only on A, B, and their derivatives up to a finite order).

2. $S_0(A,B) = AB$.

3. $S_1(A,B) - S_1(B,A) = \{A,B\}$.

We can now show that, if we specialize to the case of a symplectic manifold Σ of real dimension two, then eq. (2.2) can be used to determine a class of solutions to the problem of deformation quantization. We start by choosing, on the manifold Σ, a complex structure (this can always be done since the manifold is orientable and has dimension two). We then define

$$P_1 = \frac{1}{\varepsilon \omega^{z\overline{z}}}$$

and the higher P_n according to the recursion formula (2.1). It is then an easy matter to show that properties 1-2-3 above are satisfied. Let me just conclude this brief discussion on deformation quantization with a few remarks:

1. Using (2.1) one can show that

$$\varepsilon P_n = \frac{n}{\omega^{z\overline{z}}} + \mathrm{o}(\varepsilon)\,.$$

Therefore the n-th term in (2.2), although it is of order ε^n, also contains terms with higher powers of ε and therefore does not correspond to the n-th term in (3.2). One has to carefully keep track of powers of ε to go from the simpler and geometrically clear expression (2.2) to the standard form of deformation quantization (3.2). For example, if we denote for simplicity

$$\omega = \omega^{z\overline{z}}$$

then the first few terms in (3.2) read

$$\begin{aligned} A \star B = AB &+ \varepsilon\omega\partial A\overline{\partial}B + \frac{\varepsilon^2}{2}(\partial\omega\partial A)(\overline{\partial}\omega\overline{\partial}B) + \\ &+ \frac{\varepsilon^3}{6}\frac{1}{\omega}\left(\partial\omega\partial\omega\partial A\right)\left(\overline{\partial}\omega\overline{\partial}\omega\overline{\partial}B\right) + \\ &+ \frac{\varepsilon^3}{4}\left(\partial\overline{\partial}\omega\right)\left(\partial\omega\partial A\right)\left(\overline{\partial}\omega\overline{\partial}B\right) - \\ &- \frac{\varepsilon^3}{4}\frac{1}{\omega}(\partial\omega)(\overline{\partial}\omega)\left(\partial\omega\partial A\right)\left(\overline{\partial}\omega\overline{\partial}B\right) + o(\varepsilon^4)\,. \end{aligned} \tag{3.3}$$

2. If one chooses a different complex structure on Σ one gets a different $\star$ product. It is, on the other hand, know [19] that there is a one-to-one correspondence between Poisson structures and star-products (up to equivalences). Therefore one expects that the various products related to different complex structures on Σ should be gauge transformations (in the sense of [14]) of each-other.

3. Finally let me note that the expression (2.2) cannot be extended to the case of Poisson manifolds, as can be easily seen from eq. (3.3).

Let us now move to the actual proof of associativity of $\star$, and let us start by choosing three functions A, B, C in $\mathfrak{G}$ and by considering the product $(A \star B) \star C$, which we can write as

$$\begin{aligned}(A \star B) \star C \; &= \sum_{b,c=0}^{\infty} P_1 \cdots P_c \left(\frac{1}{P_c} \bar{\partial} \cdots \frac{1}{P_1} \bar{\partial} C \right) \times \\ &\times \left(\frac{1}{P_c} \partial \cdots \frac{1}{P_1} \partial \left(\frac{1}{P_b} \partial \cdots \frac{1}{P_1} \partial A \right) \left(P_1 \cdots P_b \frac{1}{P_b} \bar{\partial} \cdots \frac{1}{P_1} \bar{\partial} B \right) \right) .\end{aligned}$$

Let us focus our attention on a specific summand in the above expression for fixed b, c. In particular we wish to analyze in detail the expression

$$\frac{1}{P_c} \partial \cdots \underbrace{\frac{1}{P_1}} \partial \left(\frac{1}{P_b} \partial \cdots \frac{1}{P_1} \partial A \right) \left(P_1 \cdots P_b \frac{1}{P_b} \bar{\partial} \cdots \frac{1}{P_1} \bar{\partial} B \right) . \tag{3.4}$$

The first holomorphic derivative $\frac{1}{P_1}\partial$ (underlined by $\underbrace{\quad}$) can act on either the first or on the second parenthesis. In the first case we write the result as

$$\frac{1}{P_c} \partial \cdots \frac{1}{P_2} \partial \left(\frac{1}{P_{b+1}} \partial \frac{1}{P_b} \partial \cdots \frac{1}{P_1} \partial A \right) \left(P_2 \cdots P_{b+1} \frac{1}{P_b} \bar{\partial} \cdots \frac{1}{P_1} \bar{\partial} B \right) ,$$

where we have moved the factor of $\frac{1}{P_1}$ to the second parenthesis and we have multiplied and divided by P_{b+1} in order to maintain the general form of the first parenthesis. In the second case, on the other hand, we get

$$\frac{1}{P_c} \partial \cdots \frac{1}{P_2} \partial \left(\frac{1}{P_b} \partial \cdots \frac{1}{P_1} \partial A \right) \left(P_2 \cdots P_{b+1} \frac{1}{P_1 \cdots P_b} \partial P_1 \cdots P_b \frac{1}{P_b} \bar{\partial} \cdots \frac{1}{P_1} \bar{\partial} B \right) .$$

The reasons for rewriting the second parenthesis in this way will become clear later. In a similar way we can distribute the action of the various holomorphic derivatives on either the left or on the right parenthesis. A convenient way to summarize the result is as follows. Let s_i $(i = 1, \ldots, c)$, $s_i \in \{L, R\}$, be a string indicating whether the i-th holomorphic derivative $\frac{1}{P_i}\partial$ should act on the left or on the right parenthesis. Also let $L(s)$ be the total number of indices i such that $s_i = L$. Expression (3.4) can then be written as a sum over all possible choices of s_i

$$\sum_{\{s_i\}_{i=1}^{c}} \left(\frac{1}{P_{b+L(s)}} \partial \cdots \frac{1}{P_1} \partial A \right) K_{b,c}(s) ,$$

where the objects $K_{b,c}(s)$ are constructed using the following algorithm. Start with

$$P_l \cdots P_h \frac{1}{P_b} \bar{\partial} \cdots \frac{1}{P_1} \bar{\partial} B$$

with $l = 1$ and $h = b$. If $s_1 = L$, divide by P_l and multiply by P_{h+1}, and then raise both l and h by one unit. If $s_1 = R$, then act on the hole expression with ∂, divide by $P_l \cdots P_h$ and then multiply by $P_{l+1} \cdots P_h$. In this second case we just increase l by one, leaving h fixed. We then repeat this process for $s_2, \ldots, s_c$. The final result will be $K_{b,c}(s)$.[2] Let us note that, regardless of the specific choice of sequence s_i, at the end of the algorithm described above, the values of l and h will be respectively $1 + c$ and $b + L(s)$. Therefore the expression for $K_{b,c}(s)$ will be of the form

$$K_{b,c}(s) = P_{c+1} \cdots P_{b+L(s)} \times \cdots ,$$

where $\cdots$ depends on the specific s. We can now rewrite the full expression for $(A \star B) \star C$ as

$$\sum_{b,c=0}^{\infty} \sum_{\{s_i\}_{i=1}^{c}} \left(\frac{1}{P_{b+L(s)}} \partial \cdots \frac{1}{P_1} \partial A \right) \left(P_1 \cdots P_c K_{b,c}(s) \right) \left(\frac{1}{P_c} \overline{\partial} \cdots \frac{1}{P_1} \overline{\partial} C \right) .$$

In the above expression we change index by calling $a = b + L(s)$, and we finally arrive at the following formula:

$$(A \star B) \star C = \sum_{a,c=0}^{\infty} \left(\frac{1}{P_a} \partial \cdots \frac{1}{P_1} \partial A \right) T_{a,c} \left(\frac{1}{P_c} \overline{\partial} \cdots \frac{1}{P_1} \overline{\partial} C \right) ,$$

where

$$T_{a,c} = \sum_{\substack{\{s_i\}_{i=1}^{c} \\ L(s) \leq a}} P_1 \cdots P_c K_{a-L(s),c}(s) \,. \tag{3.5}$$

Let us consider the expression for $T_{a,c}$. The simplest case is when $c = 0$. It is immediate to show that

$$T_{a,0} = P_1 \cdots P_a \frac{1}{P_a} \overline{\partial} \cdots \frac{1}{P_1} \overline{\partial} B$$

and therefore that

$$T_{a,0} = P_1 \cdots P_{a-1} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} T_{a-1,0} \,.$$

The next simpl est case is when $a = 0$. Then $s_i = R$ for all i, and it is easy to show, following the algorithm described above, that

$$T_{0,c} = P_1 \cdots P_c \frac{1}{P_c} \partial \cdots \frac{1}{P_1} \partial B \,.$$

The corresponding recursion relation reads

$$T_{0,c} = P_1 \cdots P_{c-1} \partial \frac{1}{P_1 \cdots P_{c-1}} T_{0,c-1} \,.$$

[2]We are using the convention that $P_a \cdots P_a = P_a$, $P_{a+1} \cdots P_a = 1$, $P_{a+2} \cdots P_a = 1/P_{a+1}$, etc.

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We now move back to the general case, and start to analyze eq. (3.5) for $c, a > 0$. We can break the sum in (3.5) in two parts and rewrite eq. (3.5) as

$$\begin{aligned} T_{a,c} &= \sum_{\substack{\{s_i\}_{i=1}^{c} \\ L(s)\leq a}} P_1 \cdots P_c K_{a-L(s),c}(s) \\ &= \sum_{\substack{\{s_i\}_{i=1}^{c} \\ s_c=L,\, L(s)\leq a}} P_1 \cdots P_c K_{a-L(s),c}(s) + \sum_{\substack{\{s_i\}_{i=1}^{c} \\ s_c=R,\, L(s)\leq a}} P_1 \cdots P_c K_{a-L(s),c}(s)\,. \end{aligned} \tag{3.6}$$

If one follows carefully the algorithm defining the functions $K_{b,c}$'s, one can show that the first term in the above expression can be rewritten as

$$\begin{aligned} \sum_{\substack{\{s_i\}_{i=1}^{c} \\ s_c=L,\, L(s)\leq a}} P_1 \cdots P_c \frac{P_a}{P_c} K_{a-L(s),c-1}(s) &= \sum_{\substack{\{s_i\}_{i=1}^{c-1} \\ L(s)\leq a-1}} P_a P_1 \cdots P_{c-1} K_{a-L(s)-1,c-1}(s) \\ &= P_a T_{a-1,c-1}\,. \end{aligned}$$

The second term of (3.6) can, on the other hand be rewritten as

$$\begin{aligned} \sum_{\substack{\{s_i\}_{i=1}^{c} \\ s_c=R,\, L(s)\leq a}} P_1 \cdots P_c \frac{1}{P_c} \partial K_{a-L(s),c-1}(s) &= \sum_{\substack{\{s_i\}_{i=1}^{c-1} \\ L(s)\leq a}} P_1 \cdots P_{c-1} \partial \frac{1}{P_1 \cdots P_{c-1}} \times \\ &\qquad \times P_1 \cdots P_{c-1} K_{a-L(s),c-1}(s) \\ &= P_1 \cdots P_{c-1} \times \partial \frac{1}{P_1 \cdots P_{c-1}} T_{a,c-1}\,. \end{aligned}$$

Combining the above two terms, we finally arrive at the general recursion relation, valid for $c, a > 0$,

$$T_{a,c} = P_a T_{a-1,c-1} + P_1 \cdots P_{c-1} \partial \frac{1}{P_1 \cdots P_{c-1}} T_{a,c-1}\,.$$

Let us summarize what we have found up to now in the following

Claim 3 *We can rewrite the expression for $(A \star B) \star C$ as*

$$(A \star B) \star C = \sum_{a,c=0}^{\infty} \left(\frac{1}{P_a} \partial \cdots \frac{1}{P_1} \partial A \right) T_{a,c} \left(\frac{1}{P_c} \overline{\partial} \cdots \frac{1}{P_1} \overline{\partial} C \right),$$

where the functions $T_{a,c}$ satisfy the following recursion relations

$$\begin{aligned} T_{0,0} &= B\,, \\ T_{0,c} &= P_1 \cdots P_{c-1} \partial \frac{1}{P_1 \cdots P_{c-1}} T_{0,c-1} && (c > 0)\,, \\ T_{a,0} &= P_1 \cdots P_{a-1} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} T_{a-1,0} && (a > 0)\,, \\ T_{a,c} &= P_a T_{a-1,c-1} + P_1 \cdots P_{c-1} \partial \frac{1}{P_1 \cdots P_{c-1}} T_{a,c-1} && (a, c > 0)\,. \end{aligned}$$

In a completely symmetric way we can also prove the following

Claim 4 *We can rewrite the expression for $A\star(B\star C)$ as*

$$A\star(B\star C)=\sum_{a,c=0}^{\infty}\left(\frac{1}{P_a}\partial\cdots\frac{1}{P_1}\partial A\right)\widetilde{T}_{a,c}\left(\frac{1}{P_c}\overline{\partial}\cdots\frac{1}{P_1}\overline{\partial}C\right),$$

where the functions $\widetilde{T}_{a,c}$ satisfy the following recursion relations

$$\begin{aligned}
\widetilde{T}_{0,0} &= B\,,\\
\widetilde{T}_{0,c} &= P_1\cdots P_{c-1}\partial\frac{1}{P_1\cdots P_{c-1}}\widetilde{T}_{0,c-1} && (c>0)\,,\\
\widetilde{T}_{a,0} &= P_1\cdots P_{a-1}\overline{\partial}\frac{1}{P_1\cdots P_{a-1}}\widetilde{T}_{a-1,0} && (a>0)\,,\\
\widetilde{T}_{a,c} &= P_c\widetilde{T}_{a-1,c-1}+P_1\cdots P_{a-1}\overline{\partial}\frac{1}{P_1\cdots P_{a-1}}\widetilde{T}_{a-1,c} && (a,c>0)\,.
\end{aligned}$$

It is then clear that, to complete the proof of associativity of the product $\star$, it suffices to show that

$$T_{a,c}=\widetilde{T}_{a,c}$$

for all $a,c\geq 0$. We break the proof in five steps.
Step 1. We first note that the recursion relations in claims 3 and 4 immediately imply that

$$T_{a,0}=\widetilde{T}_{a,0}\,,\qquad T_{0,c}=\widetilde{T}_{0,c}$$

for any $a,c\geq 0$.
Step 2. We then prove that

$$T_{1,1}=\widetilde{T}_{1,1}\,.$$

This can be shown by simply noting that

$$T_{1,1}=P_1B+\partial T_{1,0}=P_1B+\partial\overline{\partial}B$$

and that

$$\widetilde{T}_{1,1}=P_1B+\overline{\partial}T_{0,1}=P_1B+\overline{\partial}\partial B\,.$$

Step 3. We prove now that

$$T_{a,1}=\widetilde{T}_{a,1}$$

for $a\geq 2$, by induction on a. Using the recursion relations in claim 3 we can write

$$\begin{aligned}
T_{a,1} &= P_aT_{a-1,0}+\partial T_{a,0}\\
&= P_aT_{a-1,0}+\partial P_1\cdots P_{a-1}\overline{\partial}\frac{1}{P_1\cdots P_{a-1}}T_{a-1,0}\,.
\end{aligned}$$

On the other hand the equivalent recursion relation for $\widetilde{T}_{a,1}$ can be combined with the induction hypothesis $T_{a-1,1} = \widetilde{T}_{a-1,1}$ and the fact that $T_{a-1,0} = \widetilde{T}_{a-1,0}$ to show that

$$\widetilde{T}_{a,1} = P_1 T_{a-1,0} + P_1 \cdots P_{a-1} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} T_{a-1,1} \,.$$

In the above expression we then use the recursion relation for $T_{a-1,1}$ and we get

$$\begin{aligned}
\widetilde{T}_{a,1} &= P_1 T_{a-1,0} + P_1 \cdots P_{a-1} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} (P_{a-1} T_{a-2,0} + \partial T_{a-1,0}) \\
&= P_1 T_{a-1,0} + P_{a-1} P_1 \cdots P_{a-2} \overline{\partial} \frac{1}{P_1 \cdots P_{a-2}} T_{a-2,0} + \\
&\quad + P_1 \cdots P_{a-1} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} \partial T_{a-1,0} \\
&= (P_1 + P_{a-1}) T_{a-1,0} + P_1 \cdots P_{a-1} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} \partial T_{a-1,0} \,.
\end{aligned}$$

We are now almost done. In order to show that $\widetilde{T}_{a,1} = T_{a,1}$, we first note that we can use the properties of the tensors P_n to simplify the following difference

$$\begin{aligned}
&\partial P_1 \cdots P_{a-1} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} T_{a-1,0} - P_1 \cdots P_{a-1} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} \partial T_{a-1,0} = \\
&\quad = \left[\partial, P_1 \cdots P_{a-1} \left[\overline{\partial}, \frac{1}{P_1 \cdots P_{a-1}} \right] \right] T_{a-1,0} = - \left(\partial \overline{\partial} \ln P_1 \cdots P_{a-1} \right) T_{a-1,0} = \\
&\quad = (P_1 + P_{a-1} - P_a) T_{a-1,0} \,.
\end{aligned}$$

But this shows that $\widetilde{T}_{a,1} - T_{a,1} = 0$, thus concluding the inductive step.
Step 4. In a way completely equivalent to step 3 we can also prove that

$$T_{1,c} = \widetilde{T}_{1,c}$$

for $c \geq 2$.
Step 5. Finally we will show that

$$T_{a,c} = \widetilde{T}_{a,c}$$

for $a, c \geq 2$. The proof will be again by induction on both a and c, and will be very similar to steps 3 and 3, even though it will be considerably more complex notationally.

We start by writing the recursion relation satisfied by $T_{a,c}$

$$T_{a,c} = P_a T_{a-1,c-1} + P_1 \cdots P_{c-1} \partial \frac{1}{P_1 \cdots P_{c-1}} T_{a,c-1} \,.$$

In the above expression, the induction hypothesis allows us to go from T's to $\tilde{T}$'s, and we can then use the recursion relations for $\widetilde{T}$ to write

$$\begin{aligned} T_{a,c} &= P_a P_{c-1} T_{a-2,c-2} + P_a P_1 \cdots P_{a-2} \overline{\partial} \frac{1}{P_1 \cdots P_{a-2}} T_{a-2,c-1} + \\ &\quad + P_{c-1} P_1 \cdots P_{c-2} \partial \frac{1}{P_1 \cdots P_{c-2}} T_{a-1,c-2} + \\ &\quad + P_1 \cdots P_{c-1} \partial \frac{P_1 \cdots P_{a-1}}{P_1 \cdots P_{c-1}} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} T_{a-1,c-1} \,. \end{aligned}$$

On the other hand one could start with

$$\widetilde{T}_{a,c} = P_c \widetilde{T}_{a-1,c-1} + P_1 \cdots P_{a-1} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} \widetilde{T}_{a-1,c}$$

and use the induction hypothesis and the recursion formula to write

$$\begin{aligned} \widetilde{T}_{a,c} &= P_c P_{a-1} T_{a-2,c-2} + P_c P_1 \cdots P_{c-2} \partial \frac{1}{P_1 \cdots P_{c-2}} T_{a-1,c-2} + \\ &\quad + P_{a-1} P_1 \cdots P_{a-2} \overline{\partial} \frac{1}{P_1 \cdots P_{a-2}} T_{a-2,c-1} + \\ &\quad + P_1 \cdots P_{a-1} \overline{\partial} \frac{P_1 \cdots P_{c-1}}{P_1 \cdots P_{a-1}} \partial \frac{1}{P_1 \cdots P_{c-1}} T_{a-1,c-1} \,. \end{aligned}$$

At this point we just need to consider the difference $\widetilde{T}_{a,c} - T_{a,c}$ and to show that it vanishes. The complete expression for $\widetilde{T}_{a,c} - T_{a,c}$ can be written as the sum of two parts, each of which can be simplified using the properties of the tensors P_n. On one hand we can consider the difference

$$\begin{aligned} & P_1 \cdots P_{a-1} \overline{\partial} \frac{P_1 \cdots P_{c-1}}{P_1 \cdots P_{a-1}} \partial \frac{1}{P_1 \cdots P_{c-1}} T_{a-1,c-1} - \\ & - P_1 \cdots P_{c-1} \partial \frac{P_1 \cdots P_{a-1}}{P_1 \cdots P_{c-1}} \overline{\partial} \frac{1}{P_1 \cdots P_{a-1}} T_{a-1,c-1} = \\ & \quad = -P_1 \cdots P_{c-1} \left[\partial, \frac{P_1 \cdots P_{a-1}}{P_1 \cdots P_{c-1}} \left[\overline{\partial}, \frac{P_1 \cdots P_{c-1}}{P_1 \cdots P_{a-1}} \right] \right] \frac{1}{P_1 \cdots P_{c-1}} T_{a-1,c-1} \\ & \quad = \partial \overline{\partial} \ln \left(\frac{P_1 \cdots P_{a-1}}{P_1 \cdots P_{c-1}} \right) T_{a-1,c-1} = \left(P_a - P_{a-1} - P_c + P_{c-1} \right) T_{a-1,c-1} \,. \end{aligned} \tag{3.7}$$

On the other hand, we compute the difference

$$\begin{aligned} & P_c P_{a-1} T_{a-2,c-2} + P_c P_1 \cdots P_{c-2} \partial \frac{1}{P_1 \cdots P_{c-2}} T_{a-1,c-2} + \\ & + P_{a-1} P_1 \cdots P_{a-2} \overline{\partial} \frac{1}{P_1 \cdots P_{a-2}} T_{a-2,c-1} - \\ & - P_a P_{c-1} T_{a-2,c-2} - P_a P_1 \cdots P_{a-2} \overline{\partial} \frac{1}{P_1 \cdots P_{a-2}} T_{a-2,c-1} - \end{aligned}$$

$$
\begin{aligned}
&- P_{c-1}P_1\cdots P_{c-2}\partial\frac{1}{P_1\cdots P_{c-2}}T_{a-1,c-2} = \\
&\quad = (P_c - P_{c-1})P_1\cdots P_{c-2}\partial\frac{1}{P_1\cdots P_{c-2}}T_{a-1,c-2} + \\
&\qquad + (P_{a-1} - P_a)P_1\cdots P_{a-2}\overline{\partial}\frac{1}{P_1\cdots P_{a-2}}T_{a-2,c-1} + \\
&\qquad + (P_cP_{a-1} - P_{c-1}P_{a-1} + P_{c-1}P_{a-1} - P_aP_{c-1})T_{a-2,c-2} \\
&\quad = (P_c - P_{c-1} - P_a + P_{a-1})\,T_{a-1,c-1}
\end{aligned}
$$

which exactly cancels expression (3.7), thus concluding the proof of the following

Theorem 5 *For all $a, b \geq 0$*

$$\widetilde{T}_{a,c} = T_{a,c}\,.$$

Therefore

$$(A\star B)\star C = A\star(B\star C)\,.$$

4. From star products to Bergman projections

In section 2 we started with the choice of a specific holomorphic line bundle S on Σ together with a measure C, and we constructed a star-product $\star$, which is defined only in terms of the curvature tensor R. In this section we will analyze the inverse problem of reconstructing S and C given a known R.

Let then R be given on the surface. Using Dolbeault's lemma we can find a cover $\mathcal{U}_i$ of Σ and real functions $\mathcal{L}_i$ on $\mathcal{U}_i$ such that, on the i-th patch, $R = \partial\overline{\partial}\mathcal{L}_i$. On the intersections $\mathcal{U}_i\cap\mathcal{U}_j$ we have $\mathcal{L}_j = \mathcal{L}_i + \lambda_{ij} + \overline{\lambda}_{ij}$, for some holomorphic functions λ_{ij}, which are defined up to an imaginary constant. On the triple intersections $\mathcal{U}_i\cap\mathcal{U}_j\cap\mathcal{U}_k$, we then have that $\lambda_{ij} + \lambda_{jk} = \lambda_{ik} + 2\pi i\, n_{ijk}$, where the n's are constant real numbers. Let us suppose, for now, that we can redefine the functions λ_{ij} in such a way that the numbers n_{ijk} are actually integers. In this case then, the functions $g_{ij} = e^{\lambda_{ij}}$ define a holomorphic line bundle,[3] which we will denote by T. If we recall that $R = \partial\overline{\partial}\ln C$, it is natural to let $C = e^{\mathcal{L}}$, which is a section of $T\otimes\overline{T}$, and to conclude the inverse construction by letting $S = T\otimes K^{-1}$.

Let us compute the degree of the line bundle T. Recall that $d - \Gamma\, dz = d - \partial\ln C\, dz$ defines a covariant derivative for sections of T. The first Chern class of the line bundle is then given in terms of the curvature 2-form $i\overline{\partial}\Gamma\, dz\wedge d\overline{z}$ by

$$c_1(T) = -\frac{1}{2\pi i}R\, dz\wedge d\overline{z}$$

and therefore we can compute the degree of T as

$$\deg(T) = \int_\Sigma c_1(T) = -\frac{1}{2\pi i}\int_\Sigma R\, dz\wedge d\overline{z}\,. \tag{4.1}$$

[3] We are not concerned here with questions of uniqueness.

Given a generic R, the above integral will not give an integer, and therefore the inversion problem cannot be solved. On the other hand, it is a known fact that, if the integral (4.1) does yield a number in $\mathbb{Z}$, then we can redefine the λ's introduced above (not necessarily uniquely) so that the numbers n_{ijk} are all integers, and the problem of finding C and S can be solved constructively, as I have previously shown.

5. Holomorphic curves on compact Kähler spaces

We will now start to analyze the problem of the quantization of holomorphic curves embedded in compact spaces. This section, in particular, is devoted to the general case, where the target space is a generic Kähler manifold. The discussion will serve as an introduction to the next section, where we restrict our attention to curves embedded in complex tori, and where we relate our quantization procedure to Yang-Mills configurations on the dual tori.

Let us start with a generic Riemann surface Σ of genus g, embedded holomorphically in a compact Kähler manifold M. The inclusion map $\rho : \Sigma \to M$ induces on the surface Σ a Kähler form, which we will denote by

$$\mu = \frac{i}{2} Q \, dz \wedge d\overline{z} \, .$$

The integral of μ over the surface

$$A = \int_\Sigma \mu \tag{5.1}$$

is nothing but the area of Σ considered as a submanifold of the riemannian manifold M.

We now recall the main result of [1], which can be easily extended to the present setting. As a function of the quantization parameter ε, the curvature R of the associated star-product has an asymptotic expansion of the form

$$R(\varepsilon) = -\frac{1}{\varepsilon} Q + \frac{1}{2} \partial\overline{\partial} \ln Q + \partial\overline{\partial} \mathcal{G}(\varepsilon) \, , \tag{5.2}$$

where $\mathcal{G}(\varepsilon)$ is a function on Σ expressed as a power series in ε, whose first few terms are

$$\mathcal{G}(\varepsilon) = -\frac{\varepsilon}{6} \frac{1}{Q} \partial\overline{\partial} \ln Q + \frac{\varepsilon^2}{24} \frac{1}{Q} \partial\overline{\partial} \frac{1}{Q} \partial\overline{\partial} \ln Q + \cdots \, .$$

We are now faced with the problem, analyzed in the last section, of finding the correct quantization S, C as a function of ε. We expect that not all values of ε will be allowed, since we must impose that $\int R(\varepsilon) \, dz \wedge d\overline{z} \in 2\pi i \mathbb{Z}$. On the other hand we note that, in the expansion (5.2), all the terms with positive powers of ε are total derivatives and integrate to zero. Therefore we are left with the simpler quantization condition for ε

$$\frac{1}{2\pi i}\int_\Sigma \left(-\frac{1}{\varepsilon}Q + \frac{1}{2}\partial\overline{\partial}\ln Q\right) dz \wedge d\overline{z} \in \mathbb{Z}\,.$$

We note that the second term in the integrand is proportional to the riemannian curvature on the surface, and the integral can be computed using the formula of Gauss and Bonnet. The result, $1-g$, is integral and independent of the embedding. Therefore we finally arrive at the following quantization condition for ε

$$n = \frac{1}{\pi\varepsilon}\int_\Sigma \mu\,, \qquad \varepsilon = \frac{1}{\pi}\frac{A}{n}\,.$$

To find the correct holomorphic line bundle S, we start by noticing that the 2-form μ/A can be associated, using the same reasoning as in section 4, to a holomorphic line bundle L of degree

$$\deg(L) = 1\,.$$

More precisely, we can find a covering $\mathcal{U}_i$ of Σ and real functions $\mathcal{K}_i$ on $\mathcal{U}_i$ such that $Q = \partial\overline{\partial}\mathcal{K}_i$ and such that $\mathcal{K}_j = \mathcal{K}_i + \kappa_{ij} + \overline{\kappa}_{ij}$. The line bundle L will then be defined by the transition functions $f_{ij} = e^{\frac{\pi}{A}\kappa_{ij}}$. Notice that we can rewrite (5.2) in terms of $\mathcal{K}$ as

$$R(\varepsilon) = \partial\overline{\partial}\mathcal{L}_i\,, \quad \mathcal{L}_i = -\frac{1}{\varepsilon}\mathcal{K}_i + \frac{1}{2}\ln Q + \mathcal{G}\,.$$

Therefore

$$C = e^{\mathcal{L}} = e^{-n\frac{\pi}{A}\mathcal{K}}\,\sqrt{Q}\,e^{\mathcal{G}}$$

transforms as a section of $T \otimes \overline{T}$, where

$$T = L^{-n} \otimes K^{1/2}$$

and $K^{1/2}$, the square root of the line bundle, is a choice of spin structure on the surface Σ. Finally we have that

$$S = L^n \otimes K^{1/2}\,.$$

The degree of S can be simply computed to be

$$\deg(S) = n\deg(L) + \frac{1}{2}\deg(K) = n + g - 1.$$

Finally we wish to compute the dimension

$$N(n) = \dim\mathcal{H}$$

of the space $\mathcal{H}$ of holomorphic sections of S. This can be done for large-n using the Riemann-Roch theorem. In fact, if $\deg(S) > \deg(K)$, or if $n > g-1$, then $h_1(S) = 0$, and we may write

$$\begin{aligned}\dim\mathcal{H} &= h_0(S) = h_0(S) - h_1(S) = 1 - g + \deg(S)\,,\\ N(n) &= n\,, \quad (n > g-1)\,.\end{aligned}$$

6. Traces of operators in the $\varepsilon \to 0$ limit

In the previous section we have shown how to pass from functions A on Σ to operators $\mathcal{A} = \mathcal{Q}(A)$ on $\mathcal{H}$ using Bergman quantization, so that products of operators are associated, at least asymptotically in ε, to star-products of functions. We now wish to use the differential representation of the Bergman projection to compute $\mathrm{tr}(\mathcal{A})$ in terms of the original function A.

Let h_i be an orthonormal basis of $\mathcal{H}$. We compute the trace of $\mathcal{A}$ as

$$\mathrm{tr}(\mathcal{A}) = \sum_i \langle i|\mathcal{A}|i\rangle = \sum_i \int_\Sigma \overline{h}_i(z) h_i(z) A(z)\, \Omega(z)\,.$$

Recalling the expression for the Bergman kernel $K(z,w) = \sum_i h_i(z)\overline{h}_i(w)$, we may rewrite

$$\begin{aligned}\mathrm{tr}(\mathcal{A}) &= \int_\Sigma K(z,z) A(z)\, \Omega(z) \\ &= \int_\Sigma \mu(z) A(z) \int_\Sigma K(z,w)\delta_z(w)\Omega(w)\,, \end{aligned} \tag{6.1}$$

where $\delta_z \in \mathcal{V}$ is the distribution with support at the point z defined by $F(z) = \int_\Sigma \mu(w)\delta_z(w)F(w)$, where $F \in \mathcal{V}$. Note that the second integral in expression (6.1) is nothing but the integral representation of the Bergman projection of δ_z. Therefore we conclude that

$$\mathrm{tr}(\mathcal{A}) = \int_\Sigma \mu(z) A(z) \pi(\delta_z)(z)\,.$$

Let us consider the above expression in the $\varepsilon \to 0$ limit. We wish to use the differential representation for the projection π, but in order to do so we must regularize the delta function distribution. Only at the end of the computation we can remove the regulator. To be concrete we will work in a particular coordinate system s centered at the point z and we regularize the distribution δ_z with a gaussian

$$\delta_z(s) \longrightarrow \delta_{z,\lambda}(s) = \frac{1}{Q(0)}\frac{1}{\pi\lambda} e^{-s\overline{s}/\lambda}\,.$$

The projection of $\delta_{z,\lambda}$ is explicitly given by

$$\pi(\delta_{z,\lambda})(s) = \sum_{n=0}^{\infty} (-1)^n \nabla \frac{1}{P_1} \cdots \nabla \frac{1}{P_n} P_1 \cdots P_n \frac{1}{P_n}\overline{\nabla} \cdots \frac{1}{P_1}\overline{\nabla}\delta_{z,\lambda}(s)\,, \tag{6.2}$$

where $\nabla = \partial + \Gamma$ and $\overline{\nabla} = \overline{\partial}$. Recall that we wish to consider the above expression evaluated at the point z (at the coordinate $s = 0$), in the limit $\lambda \to 0$. To extract the most singular part of the contribution we must then act with the antiholomorphic derivatives $\overline{\partial}$ on the gaussian function, since every derivative contributes an inverse power of λ. In doing so we are also left with a factor of $(-s)^n$. Therefore, in order not to get a vanishing result in the $s \to 0$ limit, we must replace ∇ with ∂, and we must act with the holomorphic derivatives on the factor $(-s)^n$. Using the fact that $\partial^n(-s)^n = (-1)^n n!$ and the fact that, to lowest order, $P_n = -nQ/\varepsilon$, we can then write

$$\pi(\delta_{z,\lambda})(0) \backsimeq \frac{1}{\pi\lambda Q}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\varepsilon}{\lambda}\right)^n\frac{1}{Q^n} = \frac{1}{\pi\lambda Q + \pi\varepsilon}\,.$$

In the limit $\lambda \to 0$ we finally get

$$\pi(\delta_z) \backsimeq \frac{1}{\pi\varepsilon}\,.$$

We have then concluded that, in the classical limit $\varepsilon \to 0$, we may compute traces using the formula

$$\mathrm{tr}(\mathcal{A}) \backsimeq \frac{1}{\pi\varepsilon}\int_\Sigma \mu A\,. \quad (\varepsilon \to 0 \text{ limit})\,. \tag{6.3}$$

One expects the general expression for the trace to be an asymptotic expansion in higher powers of ε

$$\begin{aligned}\mathrm{tr}(\mathcal{A}) &= \frac{1}{\pi}\int_\Sigma \omega(\varepsilon)A\,,\\ \omega(\varepsilon) &= \frac{1}{\varepsilon}\mu + \mu_o + \varepsilon\mu_1 + \cdots\,,\end{aligned} \tag{6.4}$$

where the 2-forms μ_i can be computed by looking at subleading terms in the expansion (6.2). One property of the forms μ_i can, on the other hand, be deduced with very little work. Consider the function 1 and the corresponding operator $Id_\mathcal{H}$. Using the results of section 5 we may write, for $n > g-1$,

$$\mathrm{tr}(Id_\mathcal{H}) = \dim(\mathcal{H}) = n = \frac{A}{\pi\varepsilon} = \frac{1}{\pi\varepsilon}\int_\Sigma \mu\,.$$

Therefore the expression (6.3) is exact in the case of the function 1,[4] and we then deduce that

$$\int_\Sigma \mu_i = 0\,, \quad (i \geq 0)$$

and therefore that the forms μ_i are all total derivatives.

7. Holomorphic curves on $\mathbb{T}_4$

In this final part of the paper we are going to use the results obtained in the previous sections to tackle the problem of quantization of holomorphic curves embedded in complex tori. In particular, for simplicity of notation, we are going to consider tori of real dimension 4, even though higher dimensional examples can be treated with exactly the same techniques.

[4] Recall that (6.4) is an asymptotic expansion in ε, and therefore it is not sensitive to the fact that $N(n) \neq n$ for $n \leq g-1$.

Let us first fix some notation. Euclidean 4-space $\mathbb{R}^4$, with the standard flat metric, will be parametrized by coordinates

$$x_a\,, \quad (a = 1, \ldots, 4)$$

and will be considered as a complex Kähler manifold $\mathbb{C}^2$, with a complex structure compatible with the metric. We will be definite and choose analytic coordinates $z_i\,(i = 1, 2)$

$$z_1 = x_1 + ix_2\,, \qquad z_2 = x_3 + ix_4$$

(the various possible choices are parametrized by $\mathrm{SO}(4)/\,\mathrm{U}(1) \times \mathrm{U}(1)$), in terms of which the Kähler form will be

$$\mu = \frac{i}{2}\partial z_1 \wedge \overline{\partial}\overline{z}_1 + \frac{i}{2}\partial z_2 \wedge \overline{\partial}\overline{z}_2 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4\,.$$

We fix, in $\mathbb{R}^4$, a lattice Γ of maximal rank generated by the basis vectors

$$a^A\,, \quad (A = 1, \ldots, 4)$$

and we denote by

$$\mathbb{T}_4 = \mathbb{R}^4/\Gamma$$

the quotient 4-torus, which inherits from $\mathbb{C}^2$ the Kähler structure. In what follows, we will also need to consider the torus

$$\tilde{\mathbb{T}}_4 = \mathbb{R}^4/\tilde{\Gamma}$$

dual to $\mathbb{T}_4$. In the above expression $\tilde{\Gamma}$ denotes the lattice dual to Γ, which is generated by the vectors

$$b_B\,, \quad (B = 1, \ldots, 4)$$

satisfying

$$a^A \cdot b_B = \delta^A_B\,.$$

Let us now consider a curve Σ embedded holomorphically in $\mathbb{T}_4$. We are clearly within the general framework described in section 5, with $M = \mathbb{T}_4$. With a slight abuse of notation we are denoting with μ both the Kähler form in the target space and the Kähler form induced on the surface Σ. We will also loosely talk about coordinate functions $X : \Sigma \to \mathbb{R}^4$, but we will have to keep in mind that they are multivalued functions, defined only up to elements of the lattice Γ

$$X \sim X + n_A a^A\,, \quad (n_A \in \mathbb{Z})\,.$$

The differentials dX_a are, on the other hand, well defined on the surface Σ, and one can write the Kähler form on Σ as

$$\mu = dX_1 \wedge dX_2 + dX_3 \wedge dX_4\,.$$

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As we described in detail in section 5, we have on the surface Σ, as a function of the quantization parameter ε, a well defined star product. In particular we notice that, although the product $X_a \star X_b$ is ill defined, the commutator $[X_a, X_b] = X_a \star X_b - X_b \star X_a$ only depends on derivatives of the coordinate functions, and therefore represents, asymptotically in ε, a function on the surface Σ. Moreover we recall from [1] that, if we define $Z_1 = X_1 + iX_2$, $Z_2 = X_3 + iX_4$, the deformation (5.2) of the curvature $R(\varepsilon)$ was chosen so that $[Z_1, Z_2] = 0$ and that $[Z_1, \overline{Z}_1] + [Z_2, \overline{Z}_2] = -\varepsilon$. Rewriting these relations in terms of the euclidean coordinates we have that

$$\begin{aligned} -2\pi i\Big([X_1, X_2] + [X_3, X_4] \Big) &= -\pi\varepsilon\,, \\ [X_1, X_3] - [X_2, X_4] = [X_1, X_4] + [X_2, X_3] &= 0\,. \end{aligned} \tag{7.1}$$

The functions X_a cannot be directly quantized, since they are multivalued on the surface Σ. To avoid this problem let us proceed formally and consider the objects

$$\mathrm{U}(y) = e^{2\pi i\, y\cdot X}\,,$$

where y are coordinates in $\mathbb{R}^4$. In the above expression, and in the ones that follow, all the products between functions should be considered as star-products. In particular

$$e^A = 1 + A + \frac{1}{2}A\star A + \frac{1}{6}A\star A\star A + \cdots\,.$$

For a generic y the objects $\mathrm{U}(y)$ do not represent well-defined functions on Σ. On the other hand, let us take the following point of view. Let us call $\mathfrak{g}$ the algebra (with respect to the product $\star$) of real functions on Σ ($\mathfrak{G} = \mathfrak{g}_{\mathbb{C}}$) and let us consider it as the Lie algebra of a group G (in the $\varepsilon \to 0$ limit, G becomes the group of μ-area preserving diffeomorphism on the surface Σ, and $\mathfrak{g}$ becomes the corresponding Lie algebra). We can then formally (since $X \notin \mathfrak{g}$) view the functions X as a constant G-gauge field on $\mathbb{R}^4$, and we can considers the objects $\mathrm{U}(y)$ as defining a gauge transformation. We may then analyze the gauge-transformed potential

$$A_a(y) = UX_aU^{-1} + \frac{i}{2\pi}U\partial_a U^{-1} \tag{7.2}$$

and we can show that, as opposed to the objects X_a and $\mathrm{U}(y)$, the A_a's are well defined functions on the surface Σ parametrized by the coordinates y on $\mathbb{R}^4$. In order to do so we first record the following identity:

$$e^Y X e^{-Y} - e^Y e^{-Y+X} = -1 + \sum_{n=1}^{\infty} d_n \mathrm{Ad}^n(Y)(X) + \; o(X^2)\,,$$

where

$$d_n = \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right), \qquad d_1 = \frac{1}{2},\; d_2 = \frac{1}{3}, \ldots\,.$$

Substituting $X \to X_a = -\frac{i}{2\pi}\partial_a(2\pi i\, y \cdot X)$ and $Y \to 2\pi i\, y \cdot X$ (note that when we take the derivative ∂_a in eq. (7.2) we are actually computing the linear term in X_a), we can use the above expression to obtain

$$A_a(y) = \sum_{n=1}^{\infty} d_n \,\mathrm{Ad}^n(2\pi i\, y \cdot X)(X_a)\,. \tag{7.3}$$

It is now apparent that the gauge fields A_a only depend on the commutator of the coordinate functions, and are therefore well-defined functions on the surface.

Up to now we have considered the functions $A_a(y)$ as G-gauge potentials over the whole euclidean 4-space $\mathbb{R}^4$. On the other hand we can easily show that the gauge transformation $\mathrm{U}(y)$ gives, in fact, a well-defined non-trivial bundle on the torus $\tilde{\mathbb{T}}_4$. To prove this fact, let us choose a point $y \in \mathbb{R}^4$ and an element $b \in \tilde{\Gamma}$. When we pass from $\mathbb{R}^4$ to the quotient $\tilde{\mathbb{T}}_4 = \mathbb{R}^4/\tilde{\Gamma}$, the fibers at y and at $y + b$ are glued using the transition function

$$U^{-1}(y)U(y+b) = e^{-2\pi i\, y\cdot X} e^{2\pi i\,(y+b)\cdot X}\,.$$

Using the Campbell-Hausdorf formula we can rewrite the above as

$$U^{-1}(y)\,\mathrm{U}(y+b) = e^{2\pi i\, b\cdot X + \text{commutators}} \tag{7.4}$$

and we therefore see that, although the single U's are not well-defined, the right hand side of (7.4) is the exponential of a function defined up $2\pi i\mathbb{Z}$, and therefore does represent a valid transition function between fibers.

We are therefore left with a non-trivial bundle on the dual torus, with a specific gauge potential A. To compute the curvature of the connection, we first note that implicit in eq. (7.2) is the relation

$$D = d - 2\pi i\, A$$

between the covariant derivative D and the connection A. Therefore the curvature 2-form $F = [D, D]$ is given explicitly by

$$F_{ab} = \partial_a A_b - \partial_b A_a - 2\pi i(A_a \star A_b - A_b \star A_a)\,. \tag{7.5}$$

It is then immediate, starting from eqs. (7.1) and applying the gauge transformation U, to conclude that the curvature F is almost-anti-self-dual (AASD from now on). More precisely we have that

$$F_{12} + F_{34} = -\pi\varepsilon\,, \qquad F_{13} - F_{24} = F_{14} + F_{23} = 0\,.$$

Up to this point, the whole discussion has been in terms of the algebra $\mathfrak{g}$ of real function on the surface. In fact, we have considered the fiber bundle on $\tilde{\mathbb{T}}_4$ as a

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principal bundle with underlying gauge group G, whose Lie algebra is $\mathfrak{g}$ itself. We may now use the quantization scheme described in the previous sections. First of all, we pass from the algebra $\mathfrak{g}$ of real functions to the algebra $\mathfrak{u}(N)$ of hermitian operators acting on $\mathcal{H}$. At the same time the group G is replaced with the group $\mathrm{U}(N)$, and the quantization of the connection A gives us a hermitian connection

$$\mathcal{A}_a = \mathcal{Q}(A_a)$$

with corresponding curvature

$$\mathcal{F}_{ab} = \mathcal{Q}(F_{ab}) = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a - 2\pi i \, [\mathcal{A}_a, \mathcal{A}_b] \, .$$

We are then left with a non-trivial $\mathrm{U}(N)$ principal bundle over the torus $\tilde{\mathbb{T}}_4$, with a specific connection whose curvature is AASD, in the sense that it satisfies

$$\mathcal{F}_{12} + \mathcal{F}_{34} = -\pi\varepsilon \, , \qquad \mathcal{F}_{13} - \mathcal{F}_{24} = \mathcal{F}_{14} + \mathcal{F}_{23} = 0 \, . \tag{7.6}$$

In order to study the properties of this principal bundle, it is natural to compute the Chern classes

$$c_1 = \mathrm{tr}(\mathcal{F}) \, , \qquad c_2 = \frac{1}{2}\, \mathrm{tr}^2(\mathcal{F}) - \frac{1}{2}\, \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) \, .$$

On the other hand, for reasons that will become clear later, before we start the computation we need to make a digression and to consider some simple topological properties of the surface Σ embedded in $\mathbb{T}_4$. Let us first introduce, on the torus $\mathbb{T}_4$, coordinates t_A defined by

$$t_A a^A_a = x_a \, , \qquad t_A = x_a b^a_A \, .$$

These coordinates run from 0 to 1 and it is very natural to use them in any topological descriptions of the torus $\mathbb{T}_4$. In particular the second integral de-Rahm cohomology group $H^2_{\text{de-Rahm}}(\mathbb{T}_4)$ is generated by the forms

$$\alpha_{AB} = dt_A \wedge dt_B \, , \quad (A < B) \, .$$

In a similar way the second integral homology group $H_2(\mathbb{T}_4)$ is generated by the Poincaré duals of the forms α_{AB}. These are the simplicies

$$\Delta^{AB} \, , \quad (A < B)$$

extending in the AB direction and satisfying

$$\int_{\Delta^{CD}} \alpha_{AB} = \delta^C_A \, \delta^D_B - \delta^D_A \, \delta^C_B \, . \tag{7.7}$$

Let us now consider $\rho(\Sigma)$ as an element of $H_2(\mathbb{T}_4)$. We can write

$$\rho(\Sigma) = \frac{1}{2} C_{AB} \Delta^{AB} \,,$$

where the coefficients C_{AB} are integral, and the equality should be understood in the sense of homology. The C's can be computed by using the formula (7.7) and by noting that

$$\int_\Sigma \rho^*(\alpha_{AB}) = \int_{\rho(\Sigma)} \alpha_{AB} = \int_{\frac{1}{2}C_{CD}\Delta^{CD}} \alpha_{AB} = C_{AB} \,.$$

Using the fact that $\rho^*(dx_a) = dX_a$ and the fact that $\rho^*(a \wedge b) = \rho^*(a) \wedge \rho^*(b)$, we conclude that

$$C_{AB} = b^a_A b^b_B \, I_{ab} \,, \qquad I_{ab} = \int_\Sigma dX_a \wedge dX_b \,.$$

The coefficients I_{ab} satisfy an important relation. If we consider the Kähler form μ as a symplectic form and we define on the surface Σ the corresponding Poisson bracket $\{A, B\}$ by

$$\{A, B\}\mu = dA \wedge dB \,,$$

it is the work of a moment to show that $\{Z_1, Z_2\} = 0$ and that $\sum_i \{Z_i, \overline{Z}_i\} = -2i$. In terms of the cartesian coordinates these relations read

$$\begin{aligned} &\{X_1, X_2\} + \{X_3, X_4\} = 1 \,, \\ &\{X_1, X_3\} - \{X_2, X_4\} = \{X_1, X_4\} + \{X_2, X_3\} = 0 \,. \end{aligned} \tag{7.8}$$

If we integrate the above equations on the hole surface against the symplectic form, and we use eq. (5.1), we immediately see that the coefficients I_{ab} satisfy the relations

$$I_{12} + I_{34} = A \,, \qquad I_{13} - I_{24} = I_{14} + I_{23} = 0 \,. \tag{7.9}$$

We have now all the elements needed for the computation of the Chern classes. We start with a general remark. It is a well known fact that the classes c_1 and c_2 are integral classes. On the other hand we see that, given the expression (7.3) for the gauge potential and the formulae (2.2), (6.4) for the star product and the trace, we expect to be able to write both c_1 and c_2 as power series in the quantization parameter ε, which we recall takes discrete values $\varepsilon = A/\pi n$. This means that any contribution to the Chern classes with a positive power of ε must vanish in cohomology, and therefore we might as well compute the Chern classes in the limit $\varepsilon \to 0$. In this limit we first of all notice that

$$[,] \longrightarrow \frac{\varepsilon}{2i}\{,\} \,.$$

This fact can then be used to simplify both the expression (7.3) for the gauge potential

$$A_a(y) \simeq \pi \frac{\varepsilon}{2} y^b \{X_b, X_a\} \tag{7.10}$$

and the formula (7.5) for the curvature

$$F_{ab} = \pi\varepsilon\{X_a, X_b\}\,. \tag{7.11}$$

Finally, recalling the result (6.3) regarding traces of operators in the $\varepsilon \to 0$ limit, we conclude that

$$\begin{aligned} c_1 &= \mathrm{tr}(\mathcal{F}) = \frac{1}{2}dy^a \wedge dy^b\, \mathrm{tr}(\mathcal{F}_{ab}) = \frac{1}{2}dy^a \wedge dy^b \int_\Sigma \{X_a, X_b\}\mu \\ &= \frac{1}{2}dy^a \wedge dy^b \int_\Sigma dX_a \wedge dX_b = \frac{1}{2}I_{ab}dy^a \wedge dy^b\,. \end{aligned}$$

First of all we notice that all of the ε dependance has vanished, as we expected. To show that the above expression actually does represent an integral form, we introduce, like in the case of the torus $\mathbb{T}_4$, coordinates s_B on $\tilde{\mathbb{T}}_4$ defined by

$$s^B b^b_B = y^b\,.$$

The integral cohomology $H^2_{\text{de-Rahm}}(\tilde{\mathbb{T}}_4)$ is then generated by the forms

$$\beta^{AB} = ds^A \wedge ds^B\,.$$

In terms of these coordinates, we can write that

$$c_1 = \frac{1}{2}I_{ab}dy^a \wedge dy^b = \frac{1}{2}I_{ab}b^a_A b^b_B ds^A \wedge ds^B = \frac{1}{2}C_{AB}\beta^{AB}\,,$$

thus proving that c_1 is an integral class, as expected.

We now move to the second Chern class. In this case the computation is much simpler for the following reason. We have seen that the curvature $\mathcal{F}$ is of order ε so that $\mathcal{F}\wedge\mathcal{F} \sim \varepsilon^2$. Traces, on the other hand, can be considered to be of order ε^{-1}, so that $\mathrm{tr}(\mathcal{F}\wedge\mathcal{F}) \sim \varepsilon$. By the previous reasoning we then expect $\mathrm{tr}(\mathcal{F}\wedge\mathcal{F})$ to vanish in cohomology, and therefore we conclude that

$$c_2 = \frac{1}{2}c_1^2\,.$$

To convince ourselves of this result, we may check that, to lowest order, $\mathrm{tr}(\mathcal{F}\wedge\mathcal{F})$ does indeed vanish. Using the expression (7.11) for the curvature, and again using the formula (6.3) for the trace, we see that

$$\begin{aligned} \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}) &\sim dy^a \wedge dy^b \wedge dy^c \wedge dy^d \int_\Sigma \mu\{X_a, X_b\}\{X_c, X_d\} \\ &\sim d^4y \int_\Sigma \mu\Big(2\{X_1, X_2\}\{X_3, X_4\} - 2\{X_1, X_3\}\{X_2, X_4\} + \\ &\qquad + 2\{X_1, X_4\}\{X_2, X_3\}\Big)\,. \end{aligned}$$

Using eqs. (7.8) we can rewrite the above as

$$d^4y\left(A - \frac{1}{2}\int_\Sigma \mu\{X_a, X_b\}^2\right).$$

To show that this final expression vanishes, we just need to show that $\frac{1}{2}\{X_a, X_b\}^2 = 1$. On the other hand, recalling that $\{A, B\} = Q^{-1}\varepsilon^{\alpha\beta}\partial_\alpha A \partial_\beta B$, we can easily check that $\det h_{\alpha\beta} = \frac{1}{2}Q^2\{X_a, X_b\}^2$, where $h_{\alpha\beta} = \partial_\alpha X_a \partial_\beta X_a$ is the induced metric on the surface. But since $\mu = \sqrt{\det h_{\alpha\beta}}dxdy = Qdxdy$, we have that $\det h_{\alpha\beta} = Q^2$, and therefore that

$$\frac{1}{2}\{X_a, X_b\}^2 = 1\,.$$

We have therefore shown that holomorphic curves embedded in compact tori have a matrix representation as $\mathrm{U}(n)$ connections over the dual tori, with AASD curvature. The underlying principal bundles are non-trivial, with vanishing instanton number and with first Chern class corresponding to the homology class of the surface embedded in the target space.

Let me describe briefly the simplest example of the formalism just outlined. We will rederive in a complex way some simple known results in order to connect the construction just described to a more familiar context. We will take $\mathbb{T}_4$ to be the unit cube, and we will let Σ be the $g = 1$ complex surface with modular parameter $\tau = i$. Let σ_1, σ_2 be the canonical coordinates on Σ and consider the embedding

$$\begin{aligned} x_1 &= a\sigma_1\,, & x_2 &= a\sigma_2\,, \quad (a, b \in \mathbb{Z})\,,\\ x_3 &= b\sigma_1\,, & x_4 &= b\sigma_2 \end{aligned}$$

so that

$$\mu = (a^2 + b^2)d\sigma_1 d\sigma_2\,, \qquad A = a^2 + b^2\,.$$

One can easily check that

$$\begin{aligned} \{X_1, X_2\} &= 1 - \{X_3, X_4\} = \frac{a^2}{A}\,,\\ \{X_1, X_3\} &= \{X_2, X_4\} = 0\,,\\ \{X_1, X_4\} &= -\{X_2, X_3\} = \frac{ab}{A}\,. \end{aligned}$$

This implies that (using eq. (7.10) and the fact that $\varepsilon = \frac{A}{\pi n}$) we may represent the embedded surface with a $\mathrm{U}(n)$ Yang-Mills linear connection on the dual torus given by

$$\begin{aligned} A_1(y) &= \frac{1}{2n}(-y_2a^2 - y_4ab)\,\mathbf{1}_{n\times n}\,,\\ A_2(y) &= \frac{1}{2n}(y_1a^2 + y_3ab)\,\mathbf{1}_{n\times n}\,,\\ A_3(y) &= \frac{1}{2n}(-y_2ab - y_4b^2)\,\mathbf{1}_{n\times n}\,,\\ A_4(y) &= \frac{1}{2n}(y_1ab + y_3b^2)\,\mathbf{1}_{n\times n}\,. \end{aligned}$$

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To conclude this section, we would like to discuss the question of stability of the AASD configurations that we have analyzed.

Purely within the context of Yang-Mills theory, we might worry that the solutions just described do not represent a local minimum of the YM action. In fact, using the AASD property of $\mathcal{F}$ it is easy to show that

$$\begin{aligned} S_{\mathrm{YM}} &= \int \mathrm{tr}(\mathcal{F}\wedge\star\mathcal{F}) = -\int \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}) + \int (\pi n\varepsilon)^2 \\ &= (\pi n\varepsilon)^2 \times \mathrm{Vol}_{\mathbb{T}_4} \sim \varepsilon^2 \,. \end{aligned}$$

On the other hand it is clear that eqs. (7.6) define a solution to the equations of motion. Since $\mathcal{F}+\star\mathcal{F} = \varepsilon\,\Omega$, where Ω is a covariantly constant 2-form, one can still use Bianchi identity $D\mathcal{F}=0$ to show that

$$D\star\mathcal{F} = 0\,.$$

To resolve this puzzle let us take the following point of view. We have seen that to each holomorphic curve $\rho:\Sigma\to\mathbb{T}_4$ we assign an element $\frac{1}{2}C_{AB}\Delta^{AB}\in H_2(\mathbb{T}_4)$. Moreover the coefficients $C_{AB} = b^a_A b^b_B\, I_{ab}$ are expressed themselves in terms of the coefficients I_{ab}, which satisfy the relations (7.9). Therefore the fact that the C's are integral imposes restrictions on the possible values of the area A of the embedded surface (for example, if $\mathbb{T}_4$ is just the cube of unit volume, then $b^a_A=\delta^a_A$, and $C=I$. But then $A = C_{12}+C_{34}$ is the sum of integer numbers, and is itself an integer). We see that the requirement of holomorphic embedding, together with the trivial fact that the homology class of Σ is integral, impose restrictions on the geometrical value of the area. We now consider the situation, in some sense dual to the one just described, of a principal $\mathrm{U}(n)$ bundle over the dual torus $\tilde{\mathbb{T}}_4$, with prescribed integral first Chern class $c_1 = \frac{1}{2}C_{AB}\beta^{AB}\in H^2(\tilde{\mathbb{T}}_4)$. Suppose that we also have a gauge potential, whose curvature 2-form is AASD. If we write $\mathcal{F}=\frac{1}{2}\mathcal{F}_{AB}\beta^{AB}$, it is clear that

$$C_{AB} = (-1)^{A+B+1}\int_{\tilde{T}_4} d^4t\ \mathrm{tr}(\mathcal{F}_{AB})\,.$$

Using the fact that $\mathcal{F}_{AB}=\mathcal{F}_{ab}b^a_A b^b_B$, we can write the above as

$$(-1)^{A+B}C_{AB} = b^a_A b^b_B\, N_{ab}\,,\quad N_{ab} = -\int_{\tilde{T}_4} d^4t\ \mathrm{tr}(\mathcal{F}_{ab})\,.$$

Using the AASD property of the connection, we see that numbers N_{ab} satisfy the relations

$$N_{12}+N_{34} = \pi\varepsilon\, n\,,\qquad N_{13}-N_{24} = N_{14}+N_{23} = 0\,.$$

Like in the case discussed above, the condition of integrality of the coefficients C_{AB}

imposes a constraint on the possible values of the parameter ε. This result resolves the question of stability, by proving that, purely in the context of Yang-Mills theory, the parameter ε must be quantized. Moreover we see that the possible values of the area and the possible values of ε are related by the formula $\pi\varepsilon\, n = A$, thus recovering in a different way the quantization condition for ε.

8. Conclusion

In this paper we analyze the problem of the matrix representation of holomorphic curves embedded in complex tori. To each analytically embedded membrane we associate a U(N) Yang Mills configuration on the dual torus which is almost-anti-self-dual. The corresponding principal bundle has vanishing instanton number and first Chern class corresponding to the homology class of the membrane embedded in the original torus. In order to tackle the problem, we extend previous results on quantization of Riemann surfaces which were derived originally in [1]. In particular we show that the proposed quantization scheme naturally leads to an associative star product over the space of functions on the surface.

We conclude by suggesting some future lines of investigation

(1) All of the results in this paper have been derived starting from the asymptotic expansion in powers of ε for the Bergman projection and for the star product. On the other hand it would be desirable to understand the non-perturbative aspects of the solution. In this context, various directions of investigation are possible. On one hand one can try to attach directly the problem of existence of an exact deformation $R(\varepsilon)$, whose asymptotic expansion is given by eq. (5.2), which preserves the commutators (7.1). On the other hand it might be more instructive to proceed in two steps. Let us suppose, as is plausible to do, that the expansion (5.2) is the perturbation expansion of some auxiliary field theory living on the surface Σ. One could then rephrase the problem of finding the non-perturbative contributions to $R(\varepsilon)$ in terms of the analysis of the non-perturbative structure of the auxiliary field theory itself.

(2) In this paper we have looked at the matrix representation of a fixed holomorphic curve of genus g. One may then speculate about the possible relations between the space of holomorphic curves on $\mathbb{T}_4$ and the space of AASD connections on the dual torus $\tilde{\mathbb{T}}_4$. It is known that the space of deformations of a holomorphic curve has dimension of order g (in what follows I will just do an order of magnitude discussion without keeping track of constants of order one). One the other hand one can consider the space of U(N) AASD connections at fixed N. The dimension of this space should be computable, and should be of order N. We recall that we can construct, given an embedded surface, an AASD connection with $N = N(n)$, where $N(n) = n$ for $n > g - 1$. On the other hand if $g \gg n$, one can show that $N(n) = 0$. We therefore see that, for fixed N, we can use surfaces of genus up to $g \sim N$. We then

see that, for surfaces of maximal genus, the dimensions of the space of holomorphic curves and of AASD connections agree. This gives hope that the relation between the two spaces just described can be made sharp.

(3) Finally one can hope to extend the results on Bergman projections and on star-products to the case of manifolds of complex dimension greater then 1. The main feature that should be retained is the geometric character of expressions (2.1) and (2.2). In fact in the $\dim_{\mathbb{C}} \Sigma = 1$ case eqs. (2.1) and (2.2) are considerably simpler then the corresponding power expansions in ε, as was noted in section 3.

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